# Spectral Sequences for Beginners

(mostly following Hutchings)

The long exact sequence of a pair allows us to compute $H_*(X)$ in terms of $H_*(A)$ and $H_*(X,A)$.

There is a similar LES for a triple. But what about quadruples, etc.? LES's don't work anymore. The answer is spectral sequences.

## Filtrations

$X$ = CW-complex.
We filter $X$ by subcomplexes: $X_0 \subseteq X_1 \subseteq \cdots$

$\rightsquigarrow$ filtration of $C_*(X)$: $F_p C_k$

$\rightsquigarrow$ associated graded modules:

$$G_p C_k = F_p C_k \big/ F_{p-1} C_k$$

examples ① $X_i$ = $i$-skeleton.
② For a fiber bundle, $X_i$ = pre-image of $i$-skeleton of the base.

# Filtered Chain Complexes

We have $\partial F_p C_k \subseteq F_p C_{k-1}$

$\rightsquigarrow$ induced $\partial : G_p C_k \longrightarrow G_p C_{k-1}$

$\rightsquigarrow$ associated graded chain complex $(G_p C_*, \partial)$

and: induced filtration on $H_*(X)$:

$$F_p H_k(X) = \{ \alpha \in H_k(X) : \exists \ x \in F_p C_k \text{ s.t. } \alpha = [x] \}$$

$\rightsquigarrow$ associated graded pieces $G_p H_k(X)$.

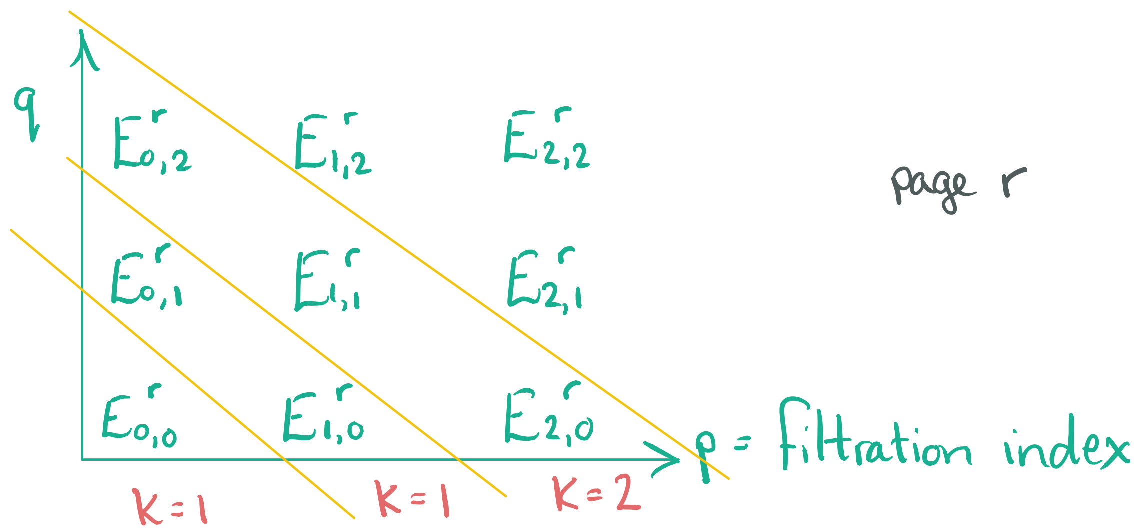
<u>Hope</u>. $H_*(G_p C_*)$ is easy to compute and it determines $G_p H_*(C_*)$, hence $H_*(X)$. We know it works for $\emptyset \subseteq A \subseteq X$.

Will compute $H_*(X)$ by "successive approximations"

# Overview

A spectral sequence has pages. Each page is a 2D grid of vector spaces (let's work over a field). There are also differentials, and we get from one page to the next by taking homology.

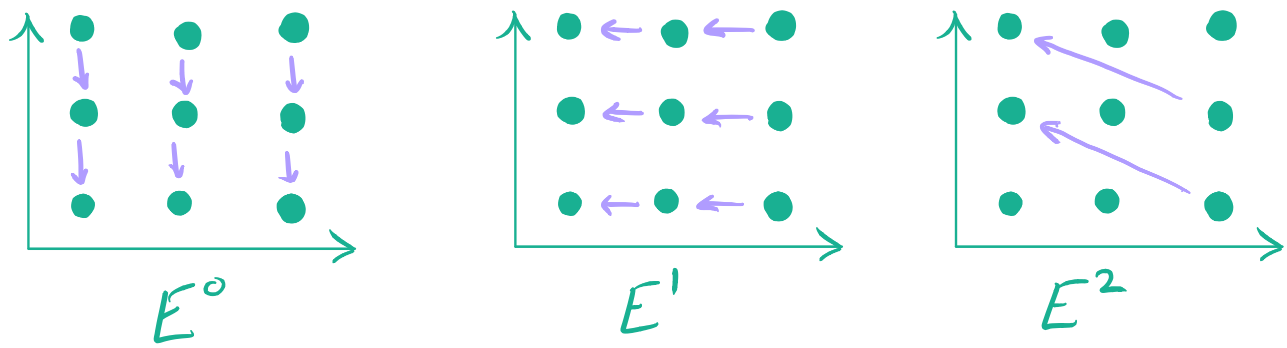
Each page looks like:




The $E^r_{p,q}$ with $p+q=k$ correspond to $k$-chains at the various levels of the filtration.

e.g. $$E^0_{p,q} = G_p C_{p+q} = F_p C_{p+q} / F_{p-1} C_{p+q}$$

The differentials always reduce dimension by 1, but as $r$ increases they go further down the filtration. Specifically, on page $r$, differentials go $r$ units left and $r-1$ units up.




In favorable cases, each term $E^r_{p,q}$ stabilizes with $r$. For instance if the $E^0_{p,q}$ are 0 outside the first quadrant (all the differentials are eventually 0). We define $E^\infty_{p,q}$ to be this term. The $\infty$ page is made of these terms.

Think about paintball. Each generator for $E^0_{p,q}$ gets a paintball. When someone shoots a paintball, both the target and the shooter get eliminated.

We will see: $E^\infty_{p,q} = G_p H_{p+q}(C_*)$

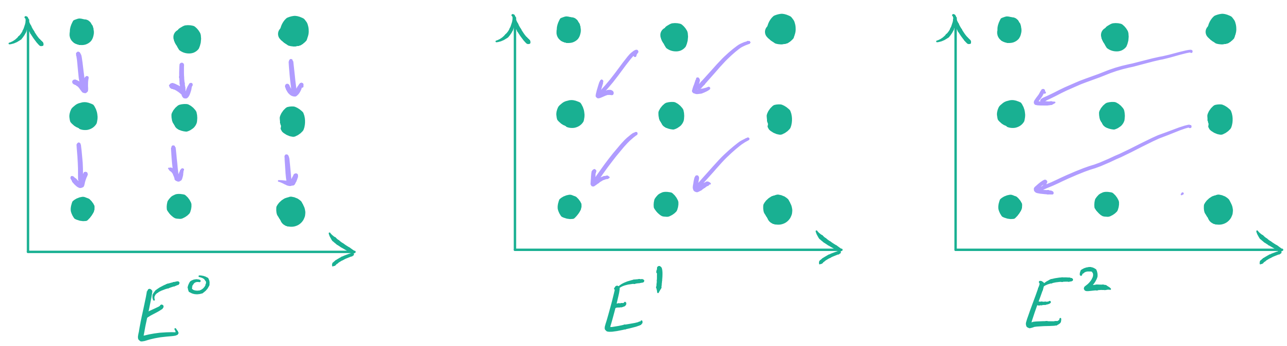
Sometimes a spectral sequence degenerates, which means all terms stabilize at the same time.

# Indexing (An Aside)

The indexing probably seems weird. Also, the way the arrows turn might seem mysterious. If we instead choose the obvious indexing:

$$E^0_{p,q} = G_p C_q$$

then the arrows are more natural:




A downside is that for most natural filtrations, the bottom right of the 1st quadrant would be 0's.

Also, Serre invented spectral sequences for fibrations. There, $E^2_{p,q} = H_p(B; H_q(F))$, which is nice!

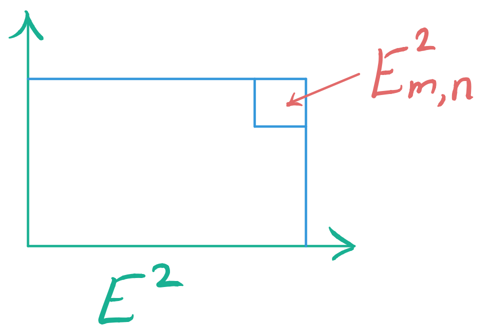
By the way, Serre's result illustrates the general pattern. If a theorem starts with "There is a spectral sequence..." then often what the theorem does is describe the $E^2$ page.

# Using Spectral Sequences

Let's say a word about using spectral sequences (yes, before we formally say what they are!)

Often, when using a long exact sequence, the hope is that there are lots of zeros. For instance, if every third term is 0, the remaining maps are isomorphisms.

It's the same with spectral sequences. Here's an example. We said that in Serre's spectral sequence we have $E^2_{p,q} = H_p(B, H_q(F))$. So if $B$ is m-dimensional and $F$ is n-dimensional, the $E^2$ page lives in the $m \times n$ rectangle:




All arrows going in & out of $E^r_{m,n}$ are 0 for $r \geq 2$.
So: $E^2_{m,n} = E^\infty_{m,n} \cong H_{m+n}(E)$.

# Formal Definitions and Statements

Say we have the $X_p$, $F_p C_*$, $G_p C_*$ as above.
We set $E^0_{p,q} = G_p C_{p+q}$
$\partial_0 : E^0_{p,q} \to E_{p,q-1}$ (= usual boundary $\partial$)

Then $E^1_{p,q}$ is obtained by taking homology at $E^0_{p,q}$, so $E^1_{p,q} = H_{p+q}(G_p C_*)$

& $\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$ is defined as:
given $\alpha \in E^1_{p,q}$, represent it by a chain
$x \in F_p C_{p+q} \rightsquigarrow \partial x \in F_p C_{p+q-1}$
$\rightsquigarrow \partial_1(\alpha) = [\partial x]$.

In other words $\partial_1$ is the usual $\partial$ in the same sense as $\delta : H_n(X,A) \longrightarrow H_{n-1}(A)$ is the usual $\partial$.

Exercise: $\partial_1$ is well def. & $\partial_1^2 = 0$.

Again, $E^2_{p,q}$ obtained by taking homology:

$$E^2_{p,q} = \frac{\ker(\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q})}{\operatorname{im}(\partial_1 : E^1_{p+1,q} \longrightarrow E^1_{p,q})}$$

In general: $E^r_{p,q} = \dfrac{\{x \in F_p C_{p+q} : \partial x \in F_{p-r} C_{p+q-1}\}}{F_{p-1} C_{p+q} + \partial(F_{p+r-1} C_{p+q+1})}$

where really we quotient by the intersection of the denominator by the numerator.

This is an approximation of cycles/boundaries: if a chain has boundary, but the boundary is far down the filtration, we consider it a cycle (for now). Similarly, if a chain is a boundary of a chain much higher in the filtration, we consider it to not be a boundary (for now).

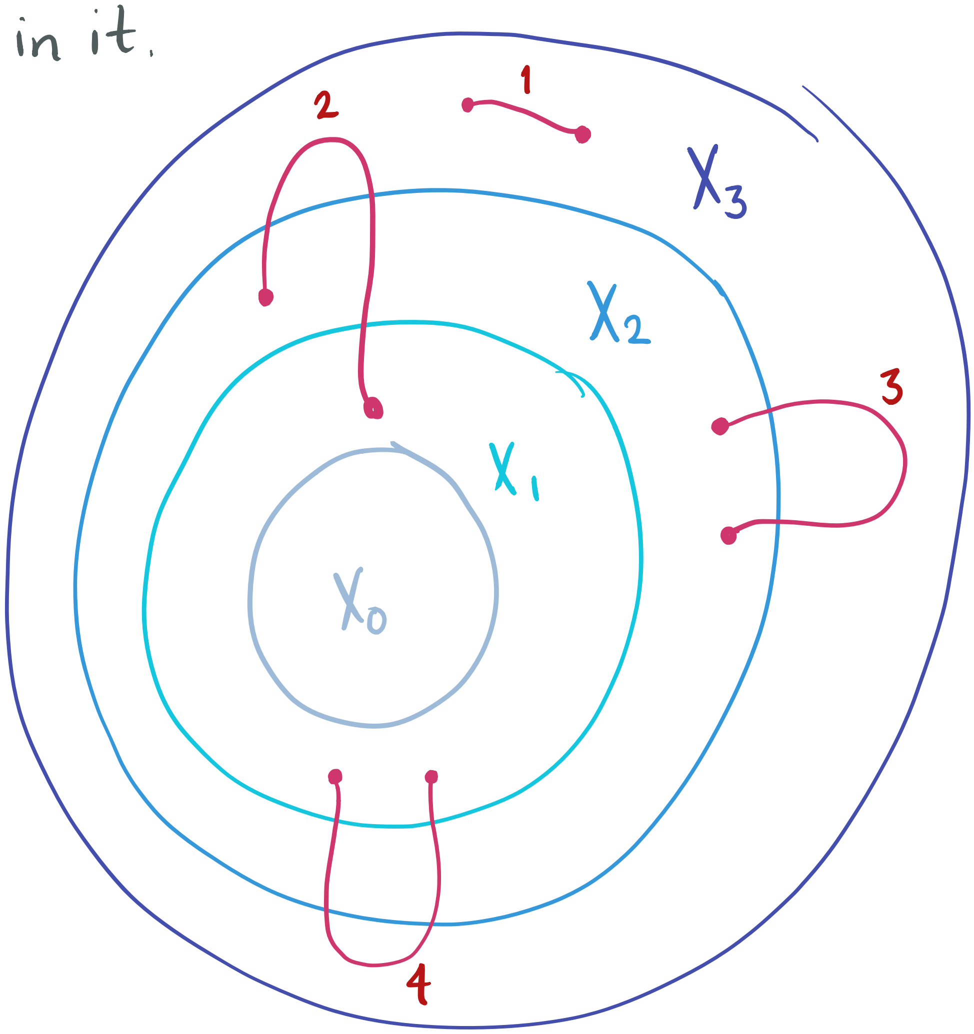
**Proposition.** Let $(F_p C_*, \partial)$ be a filtered complex, and define the $E^r_{p,q}$ as above. Then:

1. $\partial$ induces a well-defined map
$$\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1} \quad \text{with } \partial_r^2 = 0.$$
2. $E^{r+1}$ is the homology of $(E^r, \partial_r)$.
3. $E^1_{p,q} = H_{p+q}(G_p C_*)$
4. If the filtration of $C_i$ is bounded $\forall i$ then $\forall p,q$ if $r$ is sufficiently large then
$$E^r_{p,q} = G_p H_{p+q}(C_*)$$

*Pf.* Exercise

# CARTOON

Here is a schematic of a filtration, and some chains in it.




So the edge 2 lies in $X_3$, but its boundary lies in $X_2$, and one component of the boundary lies in $X_1$.

Zeroth approximation: Take boundaries in $X_p/X_{p-1}$
So a chain in $X_p$ is a cycle if its boundary lies in $X_{p-1}$. In this approximation, the edge labeled 1 is not a cycle but the others are.

First approximation: Of the remaining chains, see if they have boundary in $X_{p-1}/X_{p-2}$, etc.

The edges labeled 2 and 3 have boundary in the 1st approximation.
The edge labeled 4 has boundary in the 2nd approx.

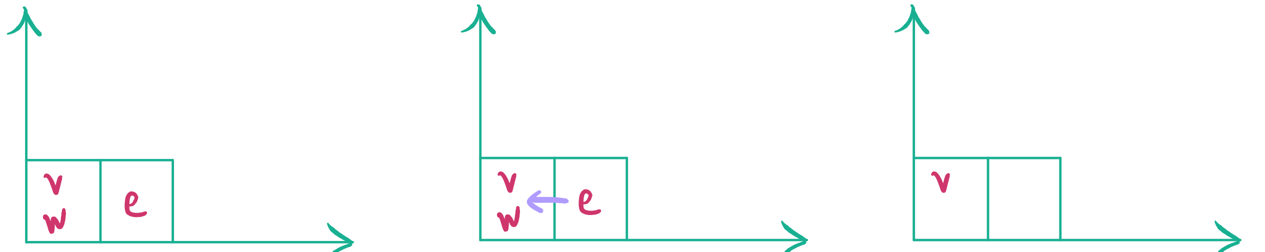
At each stage we take homology, so at the stage when we discover a chain's boundary, the boundary gets killed and the chain with boundary gets forgotten since it is not a cycle.

(Can think of searching for each chain's boundary with a stronger & stronger flashlight.)

These stages are exactly the pages of the spectral sequence.

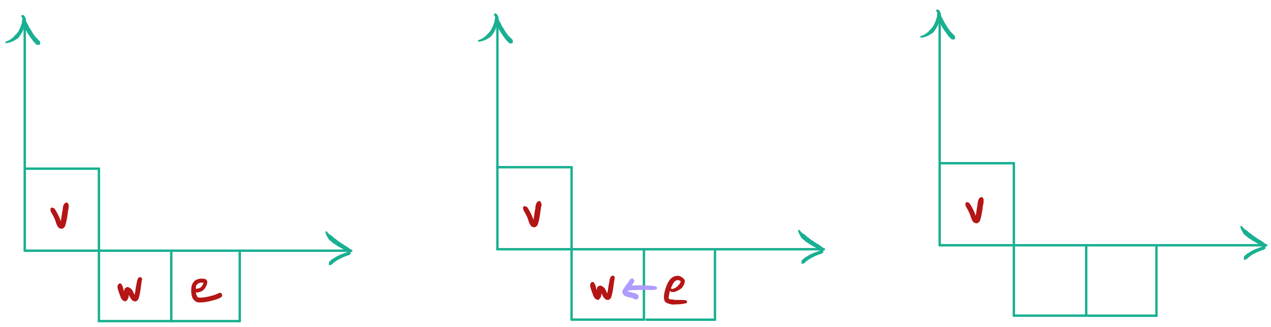
# Baby Examples

Example 1. $X = \overset{v}{\bullet}\overset{e}{\text{—}}\overset{w}{\bullet}$ , $X_0 = X^{(0)}$ , $X_1 = X^{(1)} = X$ .




$E^0$ $E^1$ $E^2 = E^\infty$

Example 2. $X = \overset{v}{\bullet}\overset{e}{\text{—}}\overset{w}{\bullet}$ , $X_0 = \{v\}$, $X_1 = \{v,w\}$, $X_2 = X$ .




$E^0$ $E^1$ $E^2 = E^\infty$

Of course we get that $H_0(X;F) = F$ both times.
The first spectral sequence gives

$$H_0(X;F) = \langle v,w\rangle / \langle v-w\rangle$$

and the second gives: $H_0(X;F) = \langle v,w\rangle / \langle w\rangle$

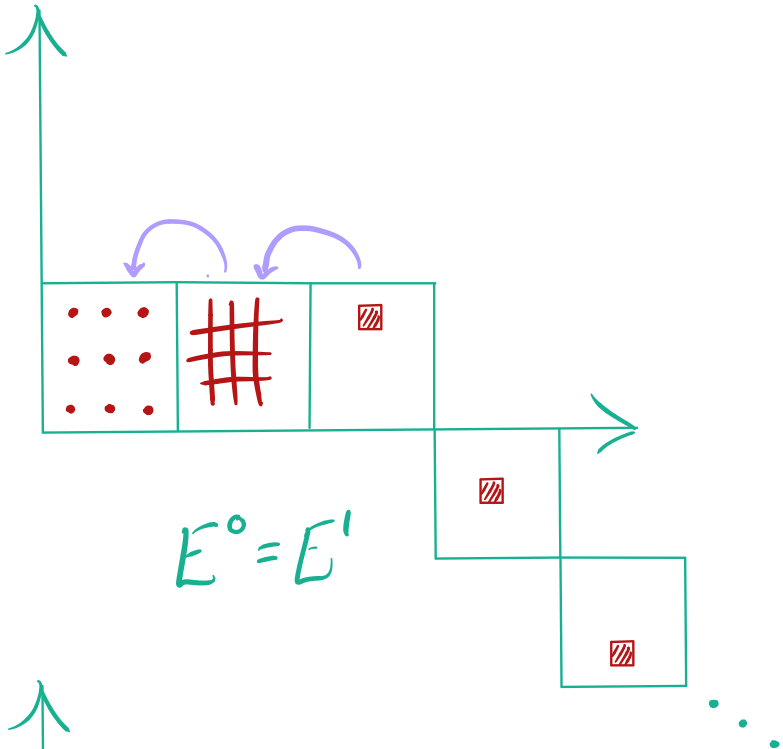
# TODDLER EXAMPLE

Example 3. $X = \mathbb{R}^2$ with usual cell decomp. into unit squares.

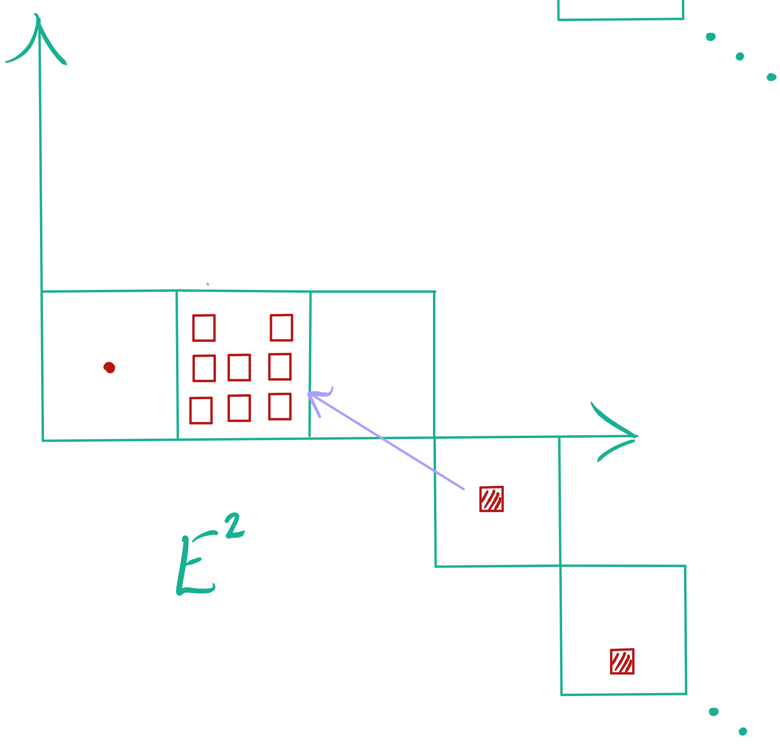
$X_0 = X^{(0)}$

$X_1 = X^{(1)}$

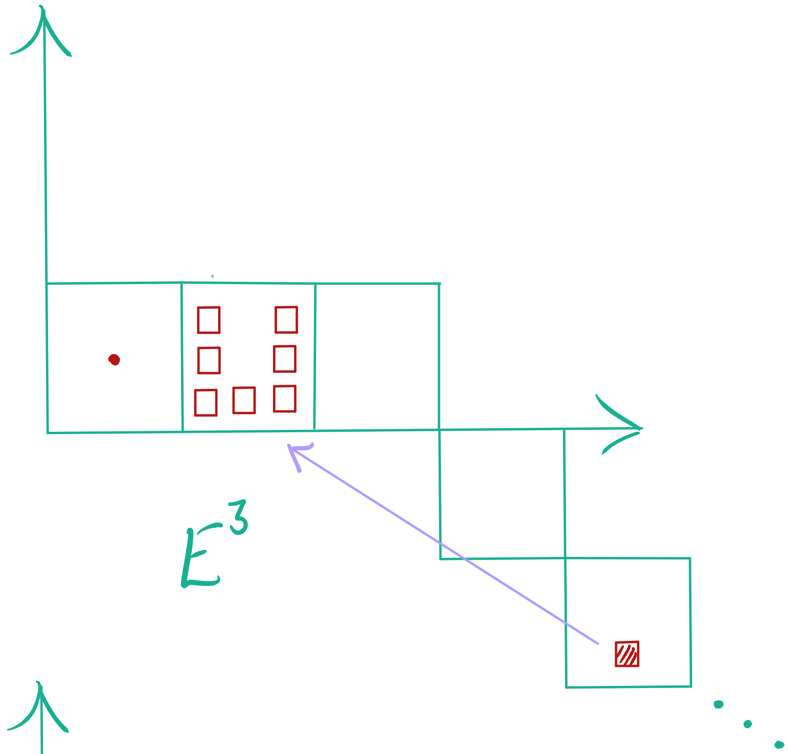
$X_i = X_{i-1} \cup \{\text{one square}\} \quad i \geq 2$

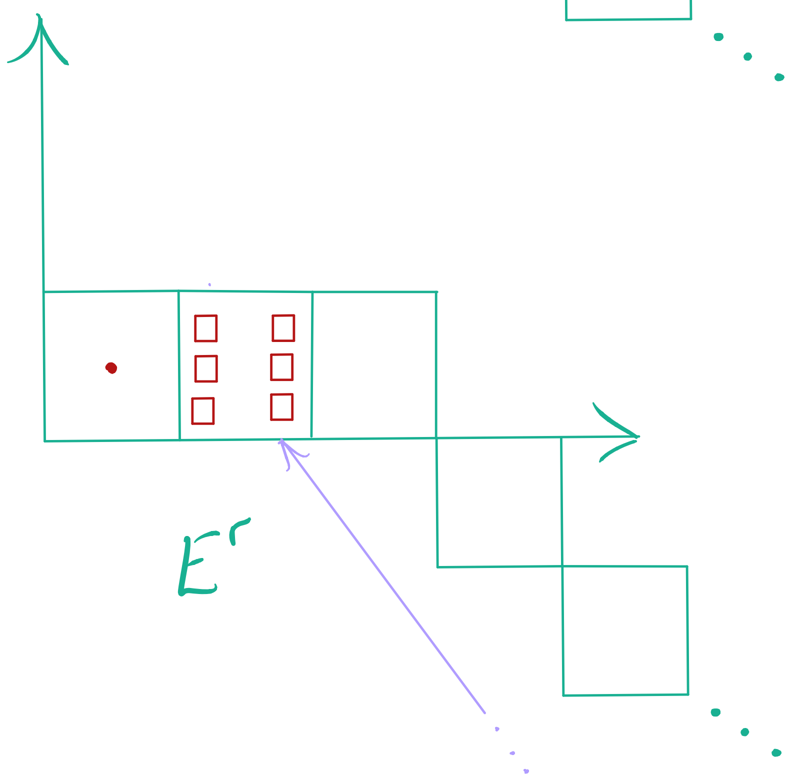


$E^0 = E^1$



$E^2$



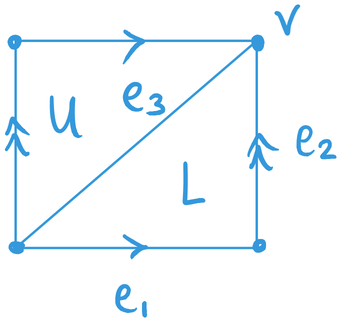




Eventually, all the squares get killed.

This filtration is not bounded, so you'll need to think about direct limits (or do a finite grid instead)

# The One-at-a-Time Spectral Sequence

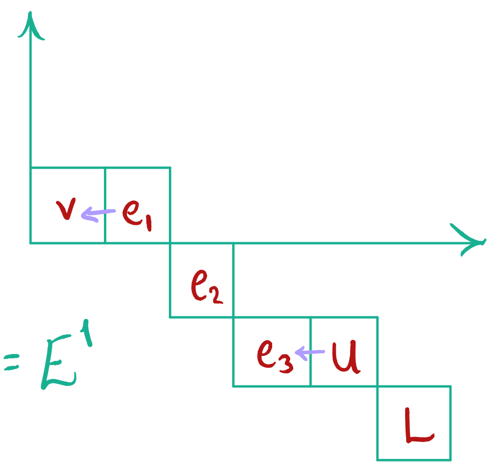
Similar to the last example, lets compute the homology of $T^2$ by adding one cell at a time. Use $\mathbb{Z}/2$

$X =$ 

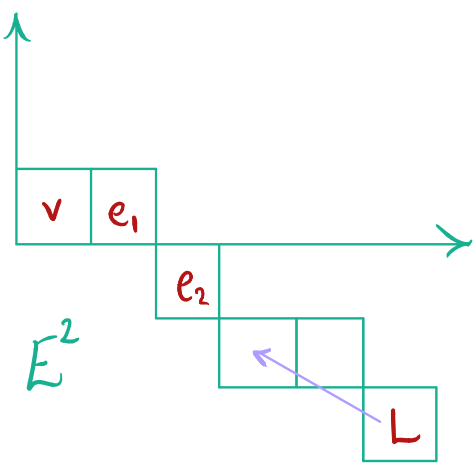

$X_0 = \{v\}$ $\quad X_3 = X_2 \cup e_3$

$X_1 = X_0 \cup e_1$ $\quad X_4 = X_3 \cup U$

$X_2 = X_1 \cup e_2$ $\quad X_5 = X_4 \cup L$




$E^0 = E^1$




$E^2$

Here we have a new phenomenon we didn't see in the last example. The cell $L$ does have boundary in $F_3 C_*$, namely $e_3$. But $e_3$ has already been eliminated. The natural way to handle this is to add $U$ to $L$, since $U$ is the cell that eliminated $e_3$. This is ok, since the $E^r_{pq}$ are all quotients. If we do this, we get that $E^2 = E^\infty$ and that $H_*(T^2)$ is generated by $v, e_1, e_2$, & $L+U$, as usual.

# APPLICATION: CELLULAR = SINGULAR

**Prop.** For $X$ a $\Delta$-complex, $H_*(X) \cong H_*^{cell}(X)$

**Pf.** Let $X_i = X^{(i)}$ (filtration by skeleta).

$\rightsquigarrow E^0_{pq} = C_{p+q}(X^{(p)})/C_{p+q}(X^{(p-1)})$

$\rightsquigarrow E^1_{pq} = H_{p+q}(X^{(p)}, X^{(p-1)})$ (by defn of rel. hom.)

Recall: $$H_{p+q}(X^{(p)}, X^{(p-1)}) \cong \begin{cases} C_p^{cell}(X) & q = 0 \\ 0 & q \neq 0 \end{cases}$$

where $C_p^{cell}(X)$ is the free $F$-module on the $p$-cells.

Now: $\partial_1 : H_p(X^{(p)}, X^{(p-1)}) \longrightarrow H_{p-1}(X^{(p)}, X^{(p-1)})$
is the usual $\partial$ (cf. LES for triple).
This exactly records the gluing maps of the $p$-cells to the $(p-1)$-skeleton.
$\Rightarrow E^2$ page is $H_*^{cell}(X)$ in bottom row, and $0$ elsewhere
$\Rightarrow E^\infty = E^2$ (the spec. seq. degenerates on page 2).
The proposition follows. $\blacksquare$

# Application: Künneth

$(C_*, \partial)$, $(C'_*, \partial')$ chain complexes over a field

$$(C \otimes C')_k = \bigoplus_{i+j=k} C_i \otimes C'_j$$

and $\partial(\alpha \otimes \beta) = (\partial\alpha) \otimes \beta + (-1)^i \alpha \otimes (\partial'\beta)$ $\quad \alpha \in C_i,\ \beta \in C'_j$

**Prop.** The natural map

$$\bigoplus_{i+j=k} H_i(C_*) \otimes H_j(C'_*) \longrightarrow H_{i+j}(C \otimes C')$$

is an isomorphism.

**Pf.** Define $F_p(C \otimes C')_k = \bigoplus_{i \leq p} C_i \otimes C'_{k-i}$

$$\rightsquigarrow E^0_{p,q} = G_p(C \otimes C')_{p+q} = C_p \otimes C'_q$$

Have
$$\begin{aligned}\partial(C_p \otimes C'_q) &\subseteq (\partial C_p \otimes C'_q) \oplus (C_p \otimes \partial' C'_q) \\ &\subseteq (C_{p-1} \otimes C'_q) \oplus (C_p \otimes C'_{q-1}) \\ &\subseteq G_{p-1} \oplus G_p\end{aligned}$$

So we already see that the spectral sequence will degenerate on page 2. The differential only reaches down one level of the filtration.

From above: $\partial_0 = (-1)^p \otimes \partial'$

We want $E^1_{pq} = {}^{\ker \partial_0}/{}_{\operatorname{im} \partial_0}$. Note the $(-1)^p$ does not affect the Kernel or the image.

$\rightsquigarrow$ $E^1_{pq}$ is the homology of the chain complex

$$\cdots \longrightarrow C_p \otimes C'_{q+1} \xrightarrow{\partial'} C_p \otimes C'_q \longrightarrow C_p \otimes C'_{q-1} \longrightarrow \cdots$$

which is, by definition: $H_*(C'_*; C_p)$.

The universal coefficient theorem for homology:

$$0 \to H_n(C'_*) \otimes C_p \longrightarrow H_n(C'_*; C_p) \longrightarrow \operatorname{Tor}(H_{n-1}(C'_*), C_p) \to 0$$

But $\operatorname{Tor}(A,B) = 0$ if $A$ or $B$ is torsion free

$$\Longrightarrow H_*(C'_*; C_p) \cong H_*(C'_*) \otimes C_p$$

So $E^1_{pq} \cong C_p \otimes H_q(C'_*)$

Next $\partial_1 = \partial \otimes 1$. Similar as above, $E^2_{pq}$ is the homology of

$$\cdots \to C_{p+1} \otimes H_q(C'_*) \longrightarrow C_p \otimes H_q(C'_*) \longrightarrow C_{p-1} \otimes H_q(C'_*) \longrightarrow \cdots$$

We are working over a field. So the $H_q(C'_*)$ are torsion free

$\rightsquigarrow$ can apply UCT as above

$\rightsquigarrow$ $E^2_{pq} = H_p(C_* \otimes H_q(C'_*)) = H_p(C_*) \otimes H_q(C'_*)$

Each elt of $E^2_{pq}$ is represented by $\alpha \otimes \beta$ where $\alpha$ is a cycle in $C_p$ & $\beta$ is a cycle in $C'_q$.

$\Rightarrow$ $\alpha \otimes \beta$ is a cycle in $C_* \otimes C'_*$.

$\Rightarrow$ all higher differentials vanish, ie. $E^2 = E^\infty$.

The proposition follows $\blacksquare$

For the Künneth formula, you also want to know that $H_*(X \times Y) \cong H_*(C_*(X) \otimes C_*(Y))$, but this is straightforward with simplicial homology.

# Fiber Bundles

Next goal: Leray-Serre spectral sequence for fiber bundles.

A fiber bundle is a space that locally looks like a product (perhaps not globally).

First examples: cylinder, Möbius band are $[0,1]$-bundles over $S^1$.

Definition. $B$ = connected space, $b_0 \in B$ base point
A continuous map $\pi: E \to B$ is a fiber bundle with fiber $F$ if
$\forall x \in B$ $\exists$ open nbd $U$ & $\psi_U$ as follows:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi_U"] \arrow[d, "\pi"'] & U \times F \arrow[dl] \\ U & \end{tikzcd}$$

Write:

$$\begin{tikzcd} \overset{\text{fiber}}{F} \arrow[r] & E \arrow[d] & \text{total space} \\ & B & \text{base} \end{tikzcd}$$

# Examples

0. Trivial bundle $E = F \times B$.
1. Covering spaces. $F =$ discrete set.
2. Cylinder & Möbius band. $F = I$, $B = S^1$
3. Torus & Klein bottle $F = S^1$, $B = S^1$
4. Vector bundles, e.g. tangent bundle
5. Sphere bundles, e.g. unit tangent bundle.
   Hopf fibration $\rightsquigarrow \pi_3(S^2) \neq 0$.
6. Mapping torus $B = S^1$.
7. Lie groups. $G =$ Lie group, $H =$ compact subgroup

$$\begin{array}{ccc} H & \longrightarrow & G \\ & & \downarrow \\ & & G/H \end{array}$$

   In fact this is a principal $H$-bundle : $H$ acts in a fiberwise way on $E = G$.
8. More Lie groups. $E =$ smooth manifold.
   $G =$ compact Lie gp
   $G \circlearrowright E$ freely, smoothly
   $\rightsquigarrow E \longrightarrow E/G$

Basic problems: classify bundles, understand sections
(Hairy ball theorem is a section problem.)

# Unitary Groups

Inner product on $\mathbb{C}^n$ : $\langle u,v\rangle = \sum u_i \bar{v}_i$

$U(n) = \{M \in GL_n\mathbb{C} : M \text{ preserves } \langle\, ,\rangle\}$

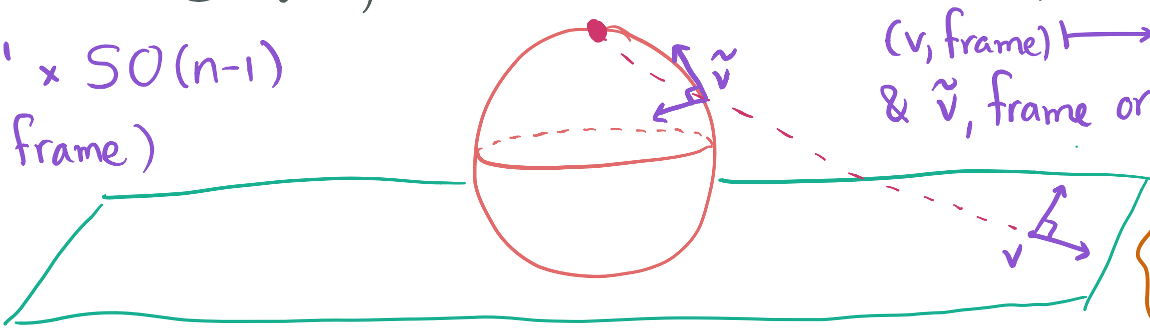
$SU(n) = \{M \in U(n) : \det(M) = 1\}$

**Prop.** We have a fiber bundle
$$\begin{array}{ccc} SU(n-1) & \longrightarrow & SU(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array}$$

**Proof #1.** $SU(n-1)$ compact subgp of Lie gp $SU(n)$.
So suffices to show $SU(n)/SU(n-1) \cong S^{2n-1}$.
$SU(n)$ acts transitively on unit sphere in $\mathbb{C}^n$, namely, $S^{2n-1}$. Stabilizer of a point is $U(n-1)$, e.g. stabilizer of $e_n$ is
$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \qquad A \in SU(n-1)$$

**Proof #2.** (O(n) version) Stereographic projection is conformal.
So the inverse maps the trivial $SO(n-1)$-bundle over $\mathbb{R}^{n-1}$ to the trivial $SO(n-1)$-bundle over $S^{n-1} \setminus$ north pole.

$\mathbb{R}^{n-1} \times SO(n-1)$
(pt, frame)




$(v, \text{frame}) \longmapsto (\tilde{v}, \text{frame})$
& $\tilde{v}$, frame orthonormal.

Same for $SU(n)$

For $n=3$:

$$\begin{array}{ccc} SU(1) & \longrightarrow & SU(2) \\ \| & & \downarrow \\ \{1\} & & S^3 \end{array}$$

$\Longrightarrow SU(2) \cong S^3$

Another way to see this:

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}$$

The equation $|\alpha|^2 + |\beta|^2 = 1$ gives unit sphere in $\mathbb{C}^2$.

Also, $SU(2) = \{\text{unit quaternions}\}$

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

We will use the Serre spectral sequence to compute $H_*(SU(n))$ for $n=3,4$. (Note $H_*(SO(n))$ is already computed in Sec. 3D of Hatcher, using an explicit cell decomposition.)

Part of the point is to show off spectral sequences as a microwave oven — often you can get something useful out with little effort or deep knowledge of the inner workings.

# Serre Spectral Sequence

**Thm.** Let $E \to B$ be a fiber bundle with fiber $F$. Then there is a spectral sequence $E^r_{pq}$ with

$$E^2_{pq} = H_p(B; \{H_q(E_x)\})$$

and converging to:

$$E^\infty_{pq} \quad G_p H_{p+q}(E)$$

for some filtration on $H_*(E)$.

**Note:** The coefficients here are local. Local coefficients are the same as constant coefficients when $\pi_1(B) = 1$.

**Local Coefficients.** $\pi = \pi_1(X)$, $M = \mathbb{Z}[\pi]$-module

$\tilde{X}$ = universal cover.

Then $H_*(X; \{M\})$ is the homology of

$$C_n(\tilde{X}) \otimes_\pi M$$

really this $\mathbb{Z}[\pi]$ but we emphasize the $\pi$

For two left modules $A, B$ over a ring $R$, $A \otimes_R B$ is the abelian group gen by $\{a \otimes b\}$ subject to distributivity and: $a \otimes b = ra \otimes rb$ (ie factor out by $R$-action).

# Application to SU(n)

Let's compute $H_*(SU(3))$.

$$\begin{tikzcd} SU(2) \arrow[r] \arrow[d, phantom, "\shortparallel"] & SU(3) \arrow[d] \\ S^3 & S^5 \end{tikzcd}$$

$$\begin{array}{c|cc} S^3 & & \\ 3 & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} & \mathbb{Q} \\ \hline & 0 & 5 \\ & & & S^5 \end{array}$$

$E^2 = E^\infty$

$$\Rightarrow H_k(SU(3)) = \begin{cases} \mathbb{Q} & k = 0,3,5,8 \\ 0 & \text{otherwise.} \end{cases}$$

$$= H_k(S^3 \times S^5)$$

...And $H_*(SU(4))$

$$\begin{tikzcd} SU(3) \arrow[r] & SU(4) \arrow[d] \\ & S^7 \end{tikzcd}$$

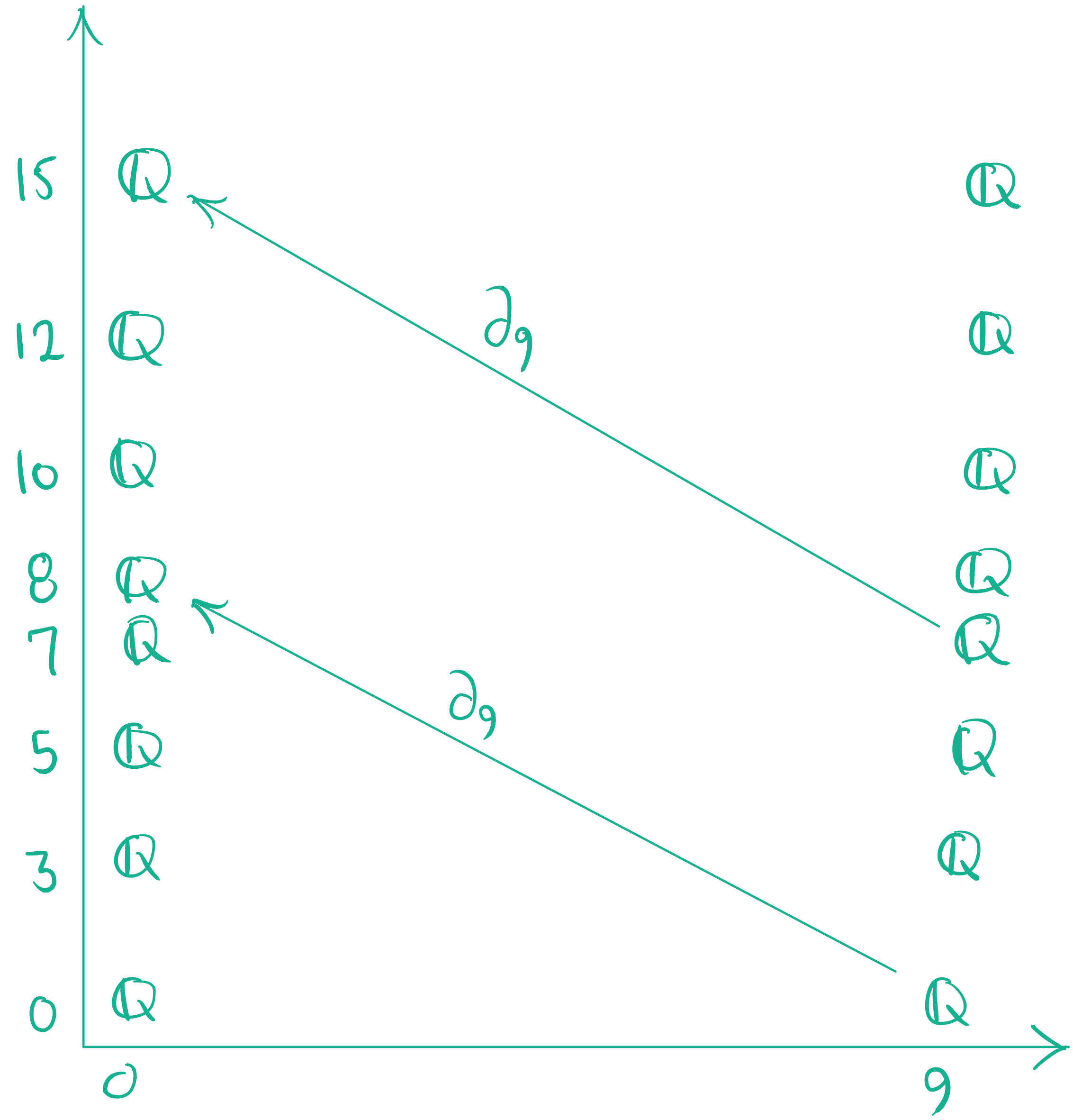
$$\begin{array}{c|cc} SU(3) & & \\ 8 & \mathbb{Q} & \mathbb{Q} \\ 5 & \mathbb{Q} & \mathbb{Q} \\ 3 & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} & \mathbb{Q} \\ \hline & 0 & 7 \\ & & & S^7 \end{array}$$

$E^2 = E^\infty$

$$\Rightarrow H_k(SU(4)) = \begin{cases} \mathbb{Q} & k = 0,3,5,8,10,12,15 \\ 0 & \text{otherwise.} \end{cases}$$

$$= H_k(S^3 \times S^5 \times S^7)$$

Unfortunately for $SU(5)$ there are differentials to consider.




But they turn out to be zero!

<u>Thm.</u> $H_*(SU(n)) = H_*(S^3 \times S^5 \times \cdots \times S^{2n-1})$

These spaces are not homotopy equivalent!

# An Example with Nontrivial Coefficients

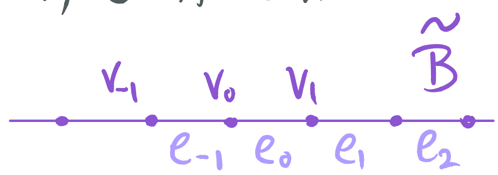
Lets compute $H_*$ of $X$ = Klein bottle with Serre:

$B = S^1 \quad F = S^1$, coefficients $M = \mathbb{Z}$ or $\mathbb{Z}/2$

$$
E^2 \qquad
\begin{array}{c|cc}
F & & \\
& H_0(B;\, H_1(F;M)) & H_1(B;\, H_1(F;M)) \\
& H_0(B;\, H_0(F;M)) & H_1(B;\, H_0(F;M)) \\
\hline
& & \qquad\qquad B
\end{array}
$$

The spectral seq. is degenerate, so it remains to compute the homology gps (and solve the extension problem).

Denote generators for $\pi_1(B)$ & $H_1(F;M)$ by $b, f$.
The action $\pi_1(B) \circlearrowleft H_k(F)$ is trivial for $k=0$
and given by $b \cdot f = -f$.
So bottom row has trivial (not local) coefficients.




Let's compute $H_*(B;\, H_1(F;M))$.
First, $C_0(\tilde{B}) \otimes H_1(F;M)$ is gen. by $v_i \otimes f$
subject to $v_i \otimes f = bv_i \otimes b\cdot f = v_{i+1} \otimes -f = -v_{i+1} \otimes f$
$\rightsquigarrow$ it is gen. by $v_0 \otimes f$

Similarly, $C_1(\tilde{B}) \otimes H_1(F;M)$ is gen by $e_0 \otimes f$

$\rightsquigarrow$ chain complex

$$0 \longrightarrow C_1(\tilde{B}) \otimes H_1(F;M) \xrightarrow{\partial \otimes 1} C_0(\tilde{B}) \otimes H_1(F;M) \longrightarrow 0$$

$$\begin{aligned} e_1 \otimes f \longmapsto (v_1 - v_0) \otimes f &= v_1 \otimes f - v_0 \otimes f \\ &= -2 v_0 \otimes f \end{aligned}$$

$$\Longrightarrow \quad H_1(B; H_1(F;\mathbb{Z})) = 0 \qquad H_0(B; H_1(F;\mathbb{Z})) = \mathbb{Z}/2$$
$$H_1(B; H_1(F;\mathbb{Z}/2)) = \mathbb{Z}/2 \qquad H_0(B; H_1(F;\mathbb{Z}/2)) = \mathbb{Z}/2.$$

$$E^2 \text{ over } \mathbb{Z}: \quad \begin{array}{|cc} \mathbb{Z}/2 & 0 \\ \mathbb{Z} & \mathbb{Z} \\ \hline \end{array} \qquad\qquad E^2 \text{ over } \mathbb{Z}/2: \quad \begin{array}{|cc} \mathbb{Z}/2 & \mathbb{Z}/2 \\ \mathbb{Z}/2 & \mathbb{Z}/2 \\ \hline \end{array}$$

This agrees with what we know:

$$H_k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z} \oplus \mathbb{Z}/2 & k=1 \\ 0 & k>1 \end{cases} \qquad H_k(X;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & k=0,2 \\ (\mathbb{Z}/2)^2 & k=1 \\ 0 & k>2 \end{cases}$$

For $H_1(X;\mathbb{Z})$ have: $0 \longrightarrow \mathbb{Z} \longrightarrow H_1(X;\mathbb{Z}) \longrightarrow \mathbb{Z}/2 \longrightarrow 0$.
Need to verify this is the trivial extension.

# Inside the Serre S.S.

Let $B^p$ = p-skeleton of B.

$F_p C_*(E)$ = singular chains supported in $\pi^{-1}(B^p)$.

$\rightsquigarrow G_p C_*(E) = C_*(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$

$\longrightarrow E^1_{pq} = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$

Can calculate as a direct sum over p-cells $\sigma : D^p \to B$ of $H_{p+q}$ of pullback bundle:

$$E^1_{pq} = \bigoplus_\sigma H_{p+q}(\sigma^* E, (\sigma^*|_{S^{p-1}})^* E)$$

pullback bundles

$$= \bigoplus_\sigma H_{p+q}(D^p \times F, S^{p-1} \times F)$$

bundles over simply connected spaces are trivial.

$$= \bigoplus_\sigma H_q(F)$$ See Fomenko p. 140.

Claim. The latter is $C_p^{cell}(B; H_q(F))$

Pf. $C_p^{cell}(B; H_q(F)) = H_p(B^p, B^{p-1}; H_q(F))$ defn.

p-dim cells of B

$$= \bigoplus_\sigma H_p(\sigma, \partial\sigma; H_q(F))$$

$$= \bigoplus H_p(\sigma/\partial\sigma; H_q(F))$$

$$= \bigoplus H_q(F). \quad \square$$

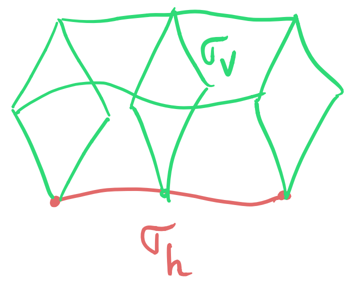
We now have $E^1$. Serre's theorem follows.

# The Serre S.S. via Cubes

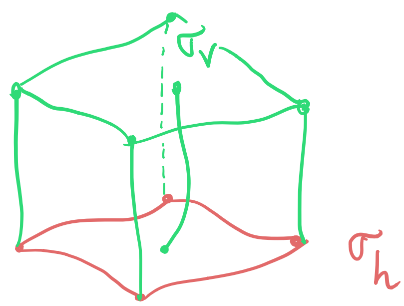
Let $C_*(E)$ be the cubical singular chain complex.

$F_p C_{p+q}(E)$ = span of the singular cubes $\sigma : I^{p+q} \longrightarrow E$ s.t. $\pi \circ \sigma$ is indep. of the last $q$ coords.

Such a cube gives a horizontal $p$-cube $\sigma_h$ and, by restricting to the center of $\sigma_h$, a vertical $q$-cube $\sigma_v$:




$F_1 C_3$




$F_2 C_3$

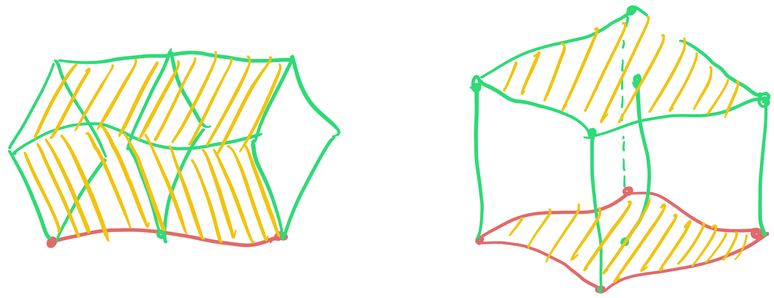
$$\rightsquigarrow \quad F_p C_{p+q}(E) \longrightarrow \bigoplus_{\sigma_h : I^p \to B} C_q(E_{\mathrm{center}(\sigma_h)})$$

$$\sigma \longmapsto (\sigma_h, \sigma_v)$$

We then mod out by degenerate $\sigma_h$, the ones indep of the last coordinate, and obtain

$$\Phi_0 : E^0_{pq} = G_p C_{p+q}(E) \longrightarrow \bigoplus_{\substack{\sigma_h : I^p \to B \\ \text{nondeg.}}} C_q(E_{\mathrm{center}(\sigma_h)})$$

The differential $\partial_0$ only considers the vertical boundary, i.e. faces obtained by forgetting one of the last $q$ coords:



So if $\Phi_0(\sigma) = (\sigma_h, \sigma_v)$ then:

$$\Phi_0(\partial\sigma) = (-1)^q (\sigma_h, \partial\sigma_v)$$

ie. fiberwise boundary.

So $\Phi_0$ induces a map on homology:

$$\Phi_1 : E^1_{pq} \longrightarrow \bigoplus_{\substack{\sigma_h : I^p \to B \\ \text{nondegen}}} H_q(E_{\text{center}(\sigma_h)}) = C_p(B; \{H_q(E_x)\})$$

Homotopy lifting property for cubes $\Longrightarrow$ $\Phi_1$ has an inverse (given $(\sigma_h, \sigma_v)$, homotope it around to get the original $\sigma$).

$\partial_1$ is the horizontal boundary. Need to use parallel transport to show this agrees with the differential on $C_p(B; \{H_q(E_x)\})$

$$\Longrightarrow E^2_{pq} = H_p(B; \{H_q(E_x)\}).$$

$\square$

# Other Spectral Sequences

Lyndon-Hochschild-Serre: Given $1 \to K \to G \to Q \to 1$

there is a spectral sequence with

$$E^2_{pq} = H_p(Q; \{H_q(K)\}) \Rightarrow H_{p+q}(G)$$

Cartan-Leray: Given $G \circlearrowright X$ free and proper

$$E^2_{pq} = H_p(G; H_q(X)) \Rightarrow H_{p+q}(X/G)$$

Or: $G \circlearrowright X$ cellularly & w/o rotations, $X \simeq *$

$$E^1_{pq} = \begin{cases} \bigoplus_{\sigma \in X_p} H_q(G_\sigma) & p,q \geq 0 \\ 0 & \text{otherwise} \end{cases} \Rightarrow H_{p+q}(G)$$

$X_p = \{p\text{-cells}\}$, $G_\sigma$ = stabilizer of $\sigma$.

... and many more (a spectral sequence for every occasion).